UDC 531.36

ASYMPTOTIC STABILITY AND INSTABILITY OF NONAUTONOMOUS SYSTEMS

PMM Vol. 43, No. 5, 1979, pp. 796 - 805

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(Received October 18, 1978)

A number of well-known theorems on asymptotic stability and instability in the presence of one Liapunov function with a sign-constant derivative are generalized to nonautonomous systems. Asymptotic stability and instability conditions are found for the equilibrium position of a holonomic mechanical system with variable masses under the action of potential, gyroscopic, and dissipative forces depending on time.

1. We consider the system of differential equations

$$x_i^{\cdot} = X_i(t, x_1, x_2, \ldots, x_n) \tag{1.1}$$
$$X_i(t, 0, 0, \ldots, 0) = 0 \quad (i = 1, 2, \ldots, n)$$

whose right hand sides are defined in the domain $G\{t \geqslant 0, \|x\| = (x_1^2 + x_2^2 + \ldots x_n^2)^{1/2} \leqslant H\}$, are bounded in this domain, $\|X(t, x)\| \leqslant Q = \text{const}$ for all $(t, x) \in G$, and satisfy the Lipschitz condition

$$\|X(t_2, x^{(2)}) - X(t_1, x^{(1)})\| \leqslant L_1(|t_2 - t_1|) + L_2(\|x_2 - x_1\|) \tag{1.2}$$
$$\forall t_2, t_1 \in [0, +\infty), \quad |t_2 - t_1| \leqslant T = \text{const}; \quad x_2, x_1 \in \Gamma\{\|x\| \leqslant H\}$$

For an arbitrary sequence of positive integers $n_r \to +\infty$ we set up a sequence of functions $X^{(r)}(\tau, x) = X((n_r - 1)T + \tau, x)$, defined in domain $G_1\{0 \leqslant \tau \leqslant T, \|x\| \leqslant H\}$. We have

$$\|X^{(r)}(\tau, x)\| \leqslant Q, \quad \|X^{(r)}(\tau_2, x_2) - X^{(r)}(\tau_1, x_1)\| = \tag{1.3}$$
$$\|X((n_r - 1)T + \tau_2, x_2) - X((n_r - 1)T + \tau_1, x_1)\| \leqslant$$
$$L_1(\|\tau_2 - \tau_1\|) + L_2(\|x_2 - x_1\|)$$

so that the sequence of functions $X^{(r)}(\tau, x)$ is uniformly bounded and equicontinuous on G_1. Consequently, according to Arzelà's theorem [1], a subsequence $n_{rs} \to +\infty$ exists such that $X^{(rs)}(\tau, x)$ converges uniformly on G_1 to some function $\varphi(\tau, x)$. The continuity of $\varphi(\tau, x)$ follows from the continuity of $X^{(r)}(\tau, x)$ and from (1.3) we get that $\varphi(\tau, x)$ satisfies a Lipschitz condition in domain G_1. Henceforth, $\varphi(t, x)$ is called the limit function of $X(t, x)$ and the set of all limit functions is denoted $N\{\varphi\}$.

We consider the system

$$x_i^{\cdot} = \varphi_i(t, x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n) \tag{1.4}$$

for which, as follows from the remarks above, the conditions for the existence and uniqueness of the solutions in G_1 are fulfilled.

Note 1.1. By the construction of system (1.4) its solution is defined in the finite time interval $[0, T]$, where the number T is specified by inequality (1.2), i.e., by the properties of the original system (1.1).

Lemma 1.1. For any solution $x = x(t, t_0, x_0)$, $t \geqslant t_0$, $x(t_0, t_0, x_0) = x_0$, of system (1.1) and for the bounded domain $\Gamma\{\|x\| \leqslant H\}$, for all $t \geqslant t_0$ there exists a sequence of segments $x^{(r)}(t) = x((n_r - 1)T + t, t_0, x_0)$, $(n_1 - 1)T \geqslant t_0$, $0 \leqslant t \leqslant T$, of this solution, converging uniformly to a function $x^*(t)$ which is a solution of system (1.4) with a preassigned limit function $\varphi = \varphi_0(t, x)$, $\varphi_0 \in N\{\varphi\}$.

Proof. By the definition of φ_0 there exists $n_r \to +\infty$ such that $X((n_r - 1)T + t, x) = X^{(r)}(t, x)$ converges uniformly to $\varphi_0(t, x)$ on G_1. We consider a sequence of segments of the solution

$$x = x(t, t_0, x_0) - x^{(r)}(t) = x((n_r - 1)T + t, t_0, x_0), \ 0 \leqslant t \leqslant T$$

with number N_0, $(N_0 - 1)T \geqslant t_0$. This sequence is uniformly bounded by virtue of the boundedness of the solution being examined and is equicontinuous by virtue of the boundedness of the derivative $dx(t, t_0, x_0)/dt$. Consequently, from the sequence $x^{(r)}(t)$ we can select a subsequence $x^{(s)}(t)$ converging uniformly to some function $x^*(t)$, $0 \leqslant t \leqslant T$. We have

$$x((n_s - 1)T + t, t_0, x_0) = x_0 + \int_{t_0}^{(n_s-1)T+t} X(\tau, x(\tau, t_0, x_0))\, d\tau =$$

$$x((n_s - 1)T, t_0, x_0) + \int_{(n_s-1)T}^{(n_s-1)T+t} X(\tau, x(\tau, t_0, x_0))\, d\tau =$$

$$x^{(s)}(0) + \int_0^t X((n_s - 1)T + \tau, x((n_s - 1)T + \tau, t_0, x_0))\, d\tau =$$

$$x^{(s)}(0) + \int_0^t [X^{(s)}(\tau, x^{(s)}(\tau)) - \varphi_0(\tau, x^{(s)}(\tau))]\, d\tau +$$

$$\int_0^t [\varphi_0(\tau, x^{(s)}(\tau)) - \varphi_0(\tau, x^*(\tau))]\, d\tau + \int_0^t \varphi_0(\tau, x^*(\tau))\, d\tau$$

Passing in this equality to the limit as $n_s \to +\infty$ and taking into account the uniform convergence of $X^{(s)}(t, x)$ to $\varphi_0(t, x)$, as well as the uniform continuity of $\varphi_0(t, x)$, we obtain

$$x^*(t) = x^*_0 + \int_0^t \varphi_0(\tau, x^*(\tau))\, d\tau \quad (0 \leqslant t \leqslant T)$$

The lemma has been proved.

The introduction of system (1.4) and Lemma 1.1 enables us to establish the asymptotic stability and instability of the zero solution of system (1.1) in the presence of one Liapunov function with a sign-constant derivative.

D e f i n i t i o n 1. 1. The set $M\{x: W(x)=0\}$ does not contain integral solutions of system (1.4) if the latter's solutions $x=x(t, x_0)$, defined on the whole finite interval $[0, T]$ and such that $x(t, x_0) \in M$ for all $0 \leqslant t \leqslant T$, do not exist.

2. T h e o r e m 2. 1. Let system (1.1) be such that:

1) a positive-definite function $V(t, x)$, $V_1(\|x\|) \leqslant V(t, x) \leqslant V_2(\|x\|)$, admitting of an infinitesimal upper bound, exists in domain G, whose total derivative relative to (1.1) is $V^{\cdot}(t, x) \leqslant W(x) \leqslant 0$;

2) a limit function $\varphi_0 \in N\{\varphi\}$ exists such that the set $M\{x: W(x)=0\}$ does not contain integral solutions of system (1.4) corresponding to this function besides the solution $x(t, 0)=0$, $0 \leqslant t \leqslant T$.

Then the zero solution of (1.1) is asymptotically stable uniformly with respect to initial coordinates from the domain

$$\Gamma_0\{\|x\| \leqslant H_0,\ H_0 < H_1 < H,\ V_2(H_0) < V_1(H_1)\}$$

P r o o f. From the theorem's hypothesis 1) follows the stability of $x=0$ uniformly with respect to t_0 [2]. Indeed, for the solution $x=x(t, t_0, x_0)$, $t_0 \geqslant 0$, $x_0 \in \Gamma_0$, of (1.1), by virtue of $V^{\cdot} \leqslant 0$ we have

$$\|x(t, t_0, x_0)\| \leqslant H_1, \quad \forall t \geqslant t_0$$
$$\lim_{t \to +\infty} V(t, x(t, t_0, x_0)) = V^* = \text{const} \geqslant 0$$

Let us assume that $V^* \neq 0$ for some such solution. We select η, $0 < \eta < V_2^{-1}(V^*)$. Then for this solution

$$\|x(t, t_0, x_0)\| \geqslant \eta, \quad \forall t \geqslant t_0 \tag{2.1}$$

Let $\varphi_0(t, x)$ be a function satisfying the theorem's hypothesis 2). By Lemma 1.1 we construct a sequence of segments $x^{(r)}(t) = x((n_r-1)T + t, t_0, x_0)$ converging to the solution $x=x^*(t)$, $0 \leqslant t \leqslant T$ of system (1.4) with function $\varphi = \varphi_0(t, x)$. By virtue of (2.1), $x^*(t) \neq 0$ for all $0 \leqslant t \leqslant T$. For the sequence $x^{(r)}(t)$ we have the estimates

$$V(n_rT, x(n_rT, t_0, x_0)) - V((n_r-1)T, x((n_r-1)T, t_0, x_0)) = \tag{2.2}$$
$$\int_{(n_r-1)T}^{n_rT} V^{\cdot} dt \leqslant -\int_{(n_r-1)T}^{n_rT} W(x(t, t_0, x_0))\, dt \leqslant -\int_0^T W(x^{(r)}(t))\, dt \leqslant 0$$

Letting $n_r \to +\infty$, we obtain

$$0 = V^* - V^* = -\int_0^T W(x^*(t))\, dt \leqslant 0$$

Hence it follows that $W(x^*(t)) = 0$ for all $0 \leqslant t \leqslant T$, which contradicts the theorem's hypothesis 2). Thus, $V^* = 0$. Consequently, from the positive-definiteness of $V(t, x)$ follows the fulfilment of the relation $\lim_{t \to +\infty} x(t, t_0, x_0) = 0$ for all $t_0 \geqslant 0$ and $x_0 \in \Gamma_0$. From the t_0-uniform stability and compactness of Γ_0 it follows [2] that the property mentioned is fulfilled uniformly with respect to $x_0 \in \Gamma_0$. This follows as well from the conditions $V^* = 0$ [3].

Theorem 2.2. Let system (1.1) be such that:

1) a function $V(t, x)$, $|V(t, x)| \leqslant V_2(\|x\|)$, admitting of an infinitesimal upper bound and having positive values in any small neighborhood of $x = 0$, exists in domain G, whose total derivative relative to (1.1) is $V^{\cdot}(t, x) \geqslant W(x) \geqslant 0$;

2) a limit function $\varphi = \varphi_0(t, x) \in N\{\varphi\}$ exists such that the set $M\{x : W(x) = 0\}$ does not contain integral solutions of system (1.4) corresponding to this functions besides the solution $x = x(t, 0)$, $0 \leqslant t \leqslant T$.

Then the zero solution is unstable.

Proof. For $t_0 \geqslant 0$ and for any small $\delta > 0$ we choose x_0, $\|x_0\| \leqslant \delta$, such that $V(t_0, x_0) = V_0 > 0$. We assume that the solution $x = x(t, t_0, x_0)$ is bounded: $\|x(t, t_0, x_0)\| \leqslant H_1 < H$ for all $t \geqslant t_0$. Then $\lim_{t\to+\infty} V(t, x(t, t_0, x_0)) = V^*$ exists by virtue of the boundedness of V. From $V^{\cdot} \geqslant 0$ we have

$$\|(x(t, t_0, x_0)\| \geqslant \eta, \quad \forall t \geqslant t_0 \tag{2.3}$$

where η is such that $V_2(\eta) < V_0$. Let $\varphi = \varphi_0(t, x)$ be a function satisfying the theorem's hypothesis 2). By Lemma 1.1. we can construct a sequence of segments $x^{(r)}(t)$ of the solution being examined, converging to the solution $x = x^*(t)$, $0 \leqslant t \leqslant T$, of system (1.4) with function $\varphi = \varphi_0(t, x)$. By virtue of (2.3), $x^*(t) \neq 0$ for all $0 \leqslant t \leqslant T$. Analogously to (2.2) we have the relations

$$V(n_rT, x(n_rT, t_0, x_0)) - V((n_r-1)T, x((n_r-1)T, t_0, x_0)) \geqslant \int_0^T W(x^{(r)}(t))\,dt \geqslant 0$$

Letting $n_r \to +\infty$, we obtain the equality

$$0 = V^* - V^* = \int_0^T W(x^*(t))\,dt \geqslant 0 \tag{2.4}$$

or $W(x^*(t)) = 0$, which contradicts the theorem's hypothesis 2).

Note 2.1. In the hypotheses of the theorems with sign-constant derivative in [4,5] we can indicate a finite number $T > 0$, such that the set $M\{x : V^{\cdot}(x) = 0\}$ does not contain the solutions

$$x(t, x_0),\ x(0, x_0) = x_0,\ x_0 \in \{x\colon 0 < \eta \leqslant \|x\| \leqslant H_1\}$$

on the whole interval $[0, T]$. Hence, taking into account that system (1.4) coincides with (1.1) when the latter is autonomous and that the number T in Theorems 2.1 and 2.2 can be arbitrary, we can show that the methods used to prove Theorems 2.1 and 2.2 are applicable to the proofs of the theorems in [4,5].

Starting from this note, we can say that Theorems 2.1 and 2.2 are generalizations of the asymptotic stability and instability theorems with sign-constant derivative in [4,5].

N o t e 2 . 2. The example in [6], showing the impossibility of a direct generalization of the theorems in [4, 5], does not satisfy the hypotheses of Theorem 2. 1.

E x a m p l e 2 . 1. *Let us consider the second-order system of differential equations*

$$x_1^{\cdot} = g_{11}(t)x_1 + g_{12}(t)x_2,\ x_2^{\cdot} = -g_{12}(t)x_1 + g_{22}(t)x_2 \tag{2.5}$$

We assume that the coefficients $g_{ij}(t)$ are bounded and satisfy the Lipschitz condition $|g_{ij}(t_2) - g_{ij}(t_1)| \leqslant L_1(|t_2 - t_1|)$, $g_{12}(t) \neq 0$ on an infinite system of intervals $[t_{1k}, t_{2k}]$, $t_{2k} - t_{1k} \geqslant T > 0$, $t_{1k} \to +\infty$ as $n_k \to +\infty$, $T = \text{const}$. Then with system (2. 5) we can associate the system

$$x_1^{\cdot} = g_{11}^*(t)x_1 + g_{12}^*(t)x_2,\ x_2^{\cdot} = -g_{12}^*(t)x_1 + g_{22}^*(t)x_2 \tag{2.6}$$

whose coefficients $g_{ij}^*(t)$ are the limits of $g_{ij}(t)$, and $g_{12}^*(t) \neq 0$ for all $0 \leqslant t \leqslant T$.

For the derivative of the function $V = x_1^2 + x_2^2$ relative to (2. 5) we have $V^{\cdot} \leqslant -hx_1^2$ if $g_{11}(t) \leqslant -h = \text{const} < 0$ and $g_{22}(t) \leqslant 0$ (case 1) or $V^{\cdot} \geqslant hx_1^2$ if $g_{11}(t) \geqslant h = \text{const} > 0$ and $g_{22}(t) \geqslant 0$ (case 2). But by virtue of the condition $g_{12}^*(t) \neq 0$ the set $M\{W(x) = \pm hx_1^2 = 0 : x_1 = 0\}$ does not contain solutions of system (2. 6) besides $x_1 = x_2 = 0$. Consequently, from Theorems 2. 1 and 2. 2 we get that the zero solution of (2. 5) is asymptotically stable in case 1) and unstable in case 2), which are weaker than the conditions required in the well-known example of Chetaev [7].

3. Let us consider the question of the asymptotic stability and instability of the zero solution of (1. 1) with respect to a part of the variables $x_1, x_2, \ldots, x_m$ $(m \leqslant n)$. For this we denote

$$y_i = x_i \quad (i = 1, 2, \ldots, m), \quad z_j = x_{m+j} \quad (j = 1, 2, \ldots, p,$$
$$p = n - m)$$
$$\|y\| = (y_1^2 + y_2^2 + \ldots + y_m^2)^{1/2}, \quad \|z\| = (z_1^2 + z_2^2 + \ldots + z_p^2)^{1/2}$$

T h e o r e m 3 . 1. Let system (1. 1) be such that:

1) its solution $x = x(t, t_0, x_0)$, $t_0 \geqslant 0$, $\|x_0\| \leqslant H_2$, is uniformly L-bounded in z, i.e., for these solutions $\|z(t, t_0, x_0)\| \leqslant L$ for all $t \geqslant t_0$ for which $\|y(t, t_0, x_0)\| \leqslant A$, $A^2 + L^2 \leqslant H_1^2$, $H_2 < H_1 < H$;

2) a y-positive-definite function $V(t, x)$, $V(t, x) \geqslant V_1(\|y\|)$, exists in domain G, such that $V(t, x) \leqslant V_2(x)$ and $V^{\cdot}(t, x) \leqslant -W(x)$; $V_2(x)$, and $W(x)$ are some sign-constant functions;

3) a limit function $\varphi = \varphi_0(t, x) \in N\{\varphi\}$ exists such that the set $M\{x : W(x) = 0\} \cap K\{x : V_2(x) > 0\}$ does not contain integral solutions of system (1. 4) corresponding to this function.

Then the zero solution of (1. 1) is y-asymptotically stable uniformly with respect to x_0 from the domain $\Gamma_0\{\|x\| \leqslant H_0 < H_2, \sup(V_2(x)$ for $x \in \Gamma_0) < V_1(A)\}$.

P r o o f. From the theorem's hypotheses 1) and 2) follows the y-stability of the zero solution of (1. 1), uniform with respect to t_0 [8]. For any solution

$x = x(t, t_0, x_0)$, $t_0 \geqslant 0$, $x_0 \in \Gamma_0$, by virtue of $V^{\cdot} \leqslant 0$ we have $\| y(t, t_0, x_0) \| \leqslant A$ for all $t \geqslant t_0$ and $\lim_{t \to +\infty} V(t, x(t, t_0, x_0)) = V^* \geqslant 0$. Thus the solution of (1.1) from Γ_0 is bounded by the domain $\Gamma_1 \{\| y \| \leqslant A, \| z \| \leqslant L\}$.

Let us assume that $V^* \neq 0$ for some solution $x = x(t, t_0, x_0)$, $x_0 \in \Gamma_0$. Then for all $t \geqslant t_0$ we have $V(t, x(t, t_0, x_0)) \geqslant V^*$, and, hence, $V_2(x(t, t_0, x_0)) \geqslant V^* \geqslant 0$. Consequently,

$$x(t, t_0, x_0) \in K_1 \{x \in \Gamma_1 : V_2(x) \geqslant V^*\}, \quad \forall t \geqslant t_0 \tag{3.1}$$

Let $\varphi_0(t, x)$ be a function satisfying the theorem's hypothesis 3). By Lemma 1.1 we construct a sequence of segments $x^{(r)} = x((n_r - 1)T + t, t_0, x_0)$ converging to the solutions $x = x^*(t)$, $0 \leqslant t \leqslant T$ of system (1.4) with function $\varphi = \varphi_0(t, x)$. By virtue of (3.1) we have $x^*(t) \in K_1$ for all $0 \leqslant t \leqslant T$. As in Theorem 2.1, from estimates (2.2) we obtain the relation $W(x^*(t)) = 0$ for all $0 \leqslant t \leqslant T$, which contradicts the theorem's hypothesis 3).

Thus, for any solution $x = x(t, t_0, x_0)$, $t_0 \geqslant 0$, $x_0 \in \Gamma_0$ we have $V^* = 0$ and, consequently, $\lim_{t \to +\infty} y(t, t_0, x_0) = 0$. According to [3], from the relation $\lim_{t \to +\infty} V(t, x(t, t_0, x_0)) = 0$ it follows as well that the asymptotic y-stability found is uniform with respect to $x_0 \in \Gamma_0$.

The nature of the instability of the zero solution of (1.1) can be established by the following theorem.

Theorem 3.2. Let system (1.1) be such that:

1) its solutions $x = x(t, t_0, x_0)$, $t_0 \geqslant 0$, $\| x_0 \| \leqslant H_0$, is uniformly L-bounded in z, i.e., for these solutions $\| z(t, t_0, x_0) \| \leqslant L$ for all $t \geqslant t_0$ such that $\| y(t, t_0, x_0) \| \leqslant A$, $H_0^2 \leqslant A^2 + L^2 = H_1^2 < H^2$;

2) a bounded function $V(t, x)$, $| V(t, x) | \leqslant V_2(\| x \|)$, exists in domain G, taking positive values in any small neighborhood of $x = 0$ whose total derivative relative to (1.1) is $V^{\cdot}(t, x) \geqslant W(x) \geqslant 0$;

3) a limit function $\varphi = \varphi_0(t, x) \in N\{\varphi\}$ exists such that the set $M\{x : W(x) = 0\} \cap K\{x : V_2(x) > 0\}$ does not contain integral solutions of system (1.4) corresponding to this function.

Then the zero solution of (1.1) is y-unstable.

Proof. For $t_0 \geqslant 0$ and any small $\delta > 0$ we choose x_0, $\| x_0 \| \leqslant \delta$, such that $V(t_0, x_0) = V_0 > 0$. We assume that the solution $x = x(t, t_0, x_0)$ is bounded in y, i.e.,

$$\| y(t, t_0, x_0) \| \leqslant A, \ A^2 + L^2 = H_1^2, \quad \forall t \geqslant t_0$$

Then by virtue of hypothesis 2), $\lim_{t \to +\infty} V(t, x(t, t_0, x_0)) = V^*$ exists and also

$$V_2(x, (t, t_0, x_0)) \geqslant V_0 > 0, \quad \forall t \geqslant t_0 \tag{3.2}$$

By Lemma 1.1 we construct a sequence of segments of the solution being examined, converging to the solution $x = x^*(t)$, $0 \leqslant t \leqslant T$, of system (1.4) with function $\varphi = \varphi_0(t, x)$. By virtue of (3.2), $V_2(x^*(t)) \geqslant V_0 > 0$; from a relation of form (2.4), as in Theorem 2.2, $W(x^*(t)) = 0$ for all $0 \leqslant t \leqslant T$. This contradicts the theorem's hypothesis 3).

Making a note analogous to Note 2.1, we can assert that Theorems 3.1 and 3.2 generalize to system (1.1) the results in [3, 9, 10] on asymptotic stability and instability with respect to a part of the variables with a sign-constant derivative.

4. Let us consider a mechanical system with variable masses and with time-independent holonomic constraints, whose motion is described by the equations [11]

$$\frac{d^*}{dt}\left(\frac{\partial T}{\partial q_i^{\cdot}}\right) - \frac{\partial^* T}{\partial q_i} = -\frac{\partial^* \Pi}{\partial q_i} + \psi_i + \sum_{j=1}^{n} g_{ij} q_j^{\cdot} - \sum_{j=1}^{n} f_{ij} q_j^{\cdot} \qquad (4.1)$$

$$T = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t, m, q) q_i^{\cdot} q_j^{\cdot}$$

Here T is the kinetic energy, $\partial^*\Pi / \partial q_i$ are generalized forces, viz., potential forces for the masses fastened, $g_{ij}(t, q_1, q_2, \ldots, q_n) = -g_{ji}(t, q_1, q_2, \ldots, q_n)$ are the coefficients of the gyroscopic forces, $f_{ij}(t, q) = f_{ji}(t, q)$ are the coefficients of the dissipative forces $\psi_i(t, q, q^{\cdot})$ are the generalized reactive forces caused by the joining and separating of particles to and from the system's material points and by the motion within these points, d^* / dt and $\partial^* / \partial q_i$ are derivatives under hardening masses. It is assumed that the masses of the points are bounded and do not vanish, i.e.,

$$0 < m_{\lambda 1} \leqslant m_{\lambda}(t, q) \leqslant m_{\lambda 2}, \quad m_{\lambda 1}, \quad m_{\lambda 2} = \text{const} \qquad (4.2)$$

$$(\lambda = 1, 2, \ldots, N)$$

In addition $\partial^*\Pi / \partial q_1 = \ldots = \partial^*\Pi / \partial q_n = 0$ for $q_1 = q_2 = \ldots = q_n = 0$ for all $t \geqslant 0$ and for any values of $m_1, \ldots, m_N$ satisfying inequalities (4.2); $\psi_1 = \psi_2 = \ldots = \psi_n = 0$ for $q_1^{\cdot} = q_2^{\cdot} = \ldots = q_n^{\cdot} = 0$, for all $t \geqslant 0$ and q; all the quantities occurring in Eqs. (4.1) satisfy conditions (1.2).

Then system (4.1) admits of the zero solution

$$q_1^{\cdot} = q_2^{\cdot} = \ldots = q_n^{\cdot} = 0, \; q_1 = q_2 = \ldots = q_n = 0 \qquad (4.3)$$

and with it we can associate the system of differential equations (Lemma 1.1)

$$\sum_{j=1}^{N} a_{ij}^{\circ} q_j^{\cdot\cdot} + \sum_{j,k=1}^{n} a_{ijk}^{\circ} q_j^{\cdot} q_k^{\cdot} - \frac{1}{2}\sum_{j,k=1}^{n} a_{jki}^{\circ} q_j^{\cdot} q_k^{\cdot} = \qquad (4.4)$$

$$\Pi_i^{\circ} + \sum_{j=1}^{n} g_{ij}^{\circ} q_j^{\cdot} - \sum f_{ij}^{\circ} q_j^{\cdot} + \psi_i^{\circ}$$

in which $a_{ij}^{\circ}, a_{ijk}^{\circ}, \Pi_i^{\circ}, g_{ij}^{\circ}, f_{ij}^{\circ}, \psi_i^{\circ}$ are the limits of the corresponding quantities from (4.1). By virtue of (4.2), $\det \| a_{ij}^{\circ} \| \geqslant \sigma = \text{const} > 0$.

Using the theorems proved, we find the conditions for the asymptotic stability and instability of solution (4.3) of system (4.1). Let us assume that system (4.1) satisfies the following constraints: the joining and separating of the particles and their motion within the material points are such that

$$\sum_{\mu=1}^{N} R_\mu v_\mu + \frac{1}{2}\sum_{\mu=1}^{N} m_\mu^{\cdot} v_\mu^2 \leqslant 0 \tag{4.5}$$

where R_μ is a reactive force applied to the μ-th material point (in the absence of internal motion of the particles [11] this condition reduces to

$$\sum_{\mu=1}^{N} m_\mu^{\cdot}\left(u_\mu - \frac{1}{2} v_\mu\right) v_\mu \leqslant 0$$

where $\vartheta_{\mu^{\cdot}}$ and u_μ are the figurative and the absolute velocities of the particles joining to and separating from the system's material points); the function $\Pi\,(t,\ m,\ q)$ does not increase with increase in time and with the change in the masses of the system's points under this increase, i.e.,

$$\frac{\partial \Pi}{\partial t} + \sum_{\mu=1}^{N} \frac{\partial \Pi}{\partial m_\mu} m_\mu^{\cdot} \leqslant 0 \tag{4.6}$$

the dissipative forces acting on the system are forces of total dissipation, and f is a positive-definite quadratic form in $q_1^{\cdot}, q_2^{\cdot}, \ldots, q_n^{\cdot}$, i.e.,

$$f = \frac{1}{2}\sum_{i,j=1}^{n} f_{ij} q_i^{\cdot} q_j^{\cdot} \geqslant f_0(q_1^{\cdot}, q_2^{\cdot}, \ldots, q_n^{\cdot}) \geqslant 0 \tag{4.7}$$
$$f_0 = 0 \Leftrightarrow q_1^{\cdot} = q_2^{\cdot} = \ldots = q_n^{\cdot} = 0$$

T h e o r e m 4. 1. Together with conditions (4.5) – (4.7) we assume as well that:

1) function $\Pi\,(t,\ m,\ q)$ is positive-definite and admits of an infinitesimal upper bound with respect to $q_1, q_2, \ldots, q_n$;

2) (4.3) is a nondegenerate isolated equilibrium position, i.e., the condition

$$\sum_{i=1}^{n}\left(\frac{\partial^* \Pi}{\partial q_i}\right)^2 \geqslant \Pi_0(q_1, q_2, \ldots, q_n) \geqslant 0, \quad \Pi_0 = 0 \Leftrightarrow q_1 = q_2 = \ldots = q_n = 0 \tag{4.8}$$

is fulfilled.

Then the zero equilibrium position of (4.1) is asymptotically stable uniformly with respect to $(q_0^{\cdot},\ q_0)$.

P r o o f. From the theorem's hypotheses the function $H = T + \Pi$ is positive-definite and admits of an infinitesimal upper bound with respect to $q_1^{\cdot}, q_2^{\cdot}, \ldots, q_n^{\cdot}, q_1, q_2, \ldots, q_n$, with total derivative

$$H^{\cdot} \leqslant -2f \leqslant -2f_0 \leqslant 0$$

where $f_0 = 0$ if and only if $q_1^{\cdot} = q_2^{\cdot} = \ldots = q_n^{\cdot} = 0$. For system (4.4) we find that its solution lying on the set $q_1^{\cdot} = q_2^{\cdot} = \ldots = q_n^{\cdot} = 0$ must satisfy the equalities $\Pi_1^{\circ} = \Pi_2^{\circ} = \ldots = \Pi_n^{\circ} = 0$, which, by virtue of (4.8), is possible only for the solution $q_1^{\cdot} = q_2^{\cdot} = \ldots = q_n^{\cdot} = q_1 = \ldots = q_n = 0$. Hence from Theorem 2.1 it follows that solution (4.3) is asymptotically stable uniformly with respect to $(q_0^{\cdot}, q_0)$.

Theorem 4.2. Together with conditions (4.5) – (4.7) we assume as well that function $\Pi = \Pi(t, m, q)$ is positive-definite in $q_1, \ldots, q_m$ and is bounded by a continuous function $P_0(q_1, q_2, \ldots, q_n)$, $\Pi(t, m, q) \leqslant P_0(q)$; the motions (4.1) from some neighborhood of (4.3) are uniformly bounded with respect to $q_{m+1}, \ldots, q_n$; there are no equilibrium positions of (4.1) on the set $P_0(q_1, q_2, \ldots, q_n) > 0$, and this property is nondegenerate, i.e.,

$$\sum_{i=1}^{n}\left(\frac{\partial^*\Pi}{\partial q_i}\right)^2 \geqslant \Pi_0(q_1, q_2, \ldots, q_n) \qquad (4.9)$$

where $\Pi_0(q_1, q_2, \ldots, q_n) > 0$ on set $P_0(q_1, q_2, \ldots q_n) > 0$. Then the zero equilibrium position of system (4.1) is asymptotically stable with respect to $q_1^{\cdot}, q_2^{\cdot}, \ldots, q_n^{\cdot}, q_1, \ldots, q_m$ uniformly with respect to $(q_0^{\cdot}, q_0)$.

Proof. From the theorem's hypotheses the function $H = T + \Pi$ is positive-definite in $q_1^{\cdot}, q_2^{\cdot}, \ldots, q_n^{\cdot}, q_1, \ldots, q_m$, $H \leqslant T_0 + P_0$, T_0 is a positive-definite quadratic form in $q_1^{\cdot}, q_2^{\cdot}, \ldots, q_n^{\cdot}$. We also have

$$H^{\cdot} \leqslant -2f, \quad M\{q^{\cdot}, q : f_0 = 0\} \cap K\{q^{\cdot}, q : T_0 + P_0 > 0\} =$$
$$\{q^{\cdot}, q : P_0(q_1, q_2, \ldots, q_n) > 0, q_1^{\cdot} = q_2^{\cdot} = \ldots = q_n^{\cdot} = 0\}$$

By virtue of (4.9),

$$\sum_{i=1}^{n}(\Pi_i^{\circ})^2 \geqslant \Pi_0(q_1, q_2, \ldots, q_n) > 0$$

on set $P_0(q_1, q_2, \ldots, q_n) > 0$. Consequently, set $M \cap K$ does not contain solutions of system (4.4). Hence, on the basis of Theorem 3.1, we obtain the proof required.

Theorem 4.3. Together with conditions (4.5) – (4.7) we assume as well that $\Pi = \Pi(t, m, q)$ admits of an infinitesimal upper bound with respect to $q_1, q_2, \ldots, q_n$; in any sufficiently small neighborhood of (4.3) the function Π takes negative values; equilibrium position (4.3) is nondegenerate and isolated, i.e., relation (4.8) holds. Then this equilibrium position is unstable.

Example 4.1. We consider a heavy rigid body of variable composition, rotating around a fixed point O, whose center of mass can be displaced along the axis Ox of some coordinate system $Oxyz$ fixed in the body; the moments of the reactive forces, of the inertial forces and of the Coriolis forces of the moving particles equal zero. The position is defined by the Euler angles θ, φ, ψ, formed by $Oxyz$ with a fixed system $O\xi\eta\zeta$; axis $O\zeta$ is directed vertically upward; $I = I(t, \theta, \varphi, \psi)$ is the body's inertia tensor in the axes $Oxyz$, dI / dt is a non-positive matrix. Besides the force of gravity with potential function $\Pi_0 = mgx_0 \sin\theta, \sin\varphi$ let the following forces act on the body: dissipative forces with total dissipation and potential forces with potential function

$$\Pi_1 = \Pi_1(t, \theta, \varphi, \psi), \quad \partial\Pi_1 / \partial t \leqslant 0$$
$$\frac{\partial\Pi_1}{\partial\theta} = \frac{\partial\Pi_1}{\partial\varphi} = \frac{\partial\Pi_1}{\partial\psi} = 0 \text{ when } \theta = \frac{\pi}{2}, \quad \varphi = \frac{3\pi}{2}, \quad \psi = 0$$

so that the equations of motion admit of the equilibrium position (the body's axis Ox is directed downward)

$$\theta^{\cdot} = \varphi^{\cdot} = \psi^{\cdot} = 0, \ \theta = \frac{\pi}{2}, \ \varphi = \frac{3\pi}{2}, \ \psi = 0 \tag{4.10}$$

If Π_1 is positive-definite, admits of an infinitesimal upper bound with respect to $\theta - \pi / 2, \varphi - 3\pi / 2, \psi$ and

$$\left(\frac{\partial\Pi_1}{\partial\theta}\right)^2 + \left(\frac{\partial\Pi_1}{\partial\varphi}\right)^2 + \left(\frac{\partial\Pi_1}{\partial\psi}\right)^2 \geqslant \Pi_{10}(\theta, \varphi, \psi) \geqslant 0$$

$$\Pi_{10} = 0 \Leftrightarrow \theta = \frac{\pi}{2}, \quad \varphi = \frac{3\pi}{2}, \quad \psi = 0$$

$$mg\, x_0 \geqslant q = \text{const} > 0, \quad (mgx_0)^{\cdot} \leqslant 0 \tag{4.11}$$

then equilibrium position (4.10) is asymptotically stable uniformly with respect to $\theta_0^{\cdot}, \varphi_0^{\cdot}, \psi_0^{\cdot}, \theta_0, \varphi_0, \psi_0$. If $\Pi_1 = \text{const}$, then under conditions (4.11) position (4.10) is asymptotically stable with respect to $\theta^{\cdot}, \varphi^{\cdot}, \psi^{\cdot}, \theta, \varphi$. When $\Pi_1 = \text{const}$ there also is the equilibrium position (the body's axis Ox is directed upward)

$$\theta^{\cdot} = \varphi^{\cdot} = \psi^{\cdot} = 0, \ \theta = \frac{\pi}{2}, \ \varphi = \frac{3\pi}{2}, \ \psi = 0$$

Under the action of dissipative forces with total dissipation and under the conditions $0 < q_1 \leqslant (mgx_0) \leqslant q_2$ ($q_1, q_2 = \text{const}$) and $(mgx_0)^{\cdot} \geqslant 0$ this equilibrium position is unstable.

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Translated by N. H. C.